

# EXISTENCE OF POSITIVE SOLUTIONS TO A SEMI-LINEAR ELLIPTIC SYSTEM WITH A SMALL ENERGY/CHARGE RATIO

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**ABSTRACT.** We prove the existence of positive solutions to a system of  $k$  non-linear elliptic equations corresponding to standing-wave  $k$ -uples solutions to a system of non-linear Klein-Gordon equations. Our solutions are characterised by a small energy/charge ratio, appropriately defined.

## INTRODUCTION

Given the real numbers  $0 < m_1 \leq m_2 \leq \dots \leq m_k$ , we show the existence of solutions to the non-linear elliptic system

$$(E) \quad -\Delta u_j + (m_j^2 - \omega_j^2)u_j + \partial_{z_j} G(u) = 0, \quad 1 \leq j \leq k$$

$$u_j > 0, \quad u_j \in H_r^1(\mathbb{R}^n)$$

which are critical points of the energy functional

$$E: H \times \Sigma \rightarrow \mathbb{R},$$

$$(u, \omega) \mapsto \frac{1}{2} \sum_{j=1}^k \int_{\mathbb{R}^n} |Du_j|^2 + (m_j^2 + \omega_j^2)u_j^2 + 2k^{-1}G(u)$$

on the constraint

$$M_\sigma := \{(u, \omega) \in H \times \Sigma \mid C_j(u, \omega) = \sigma_j\}$$

$$C_j(u, \omega) = \omega_j \int_{\mathbb{R}^n} u_j^2$$

for some  $\sigma \in (0, +\infty)^k$ . We used the notation

$$H := H^1(\mathbb{R}^n, \mathbb{R}^k), \quad \Sigma := [0, +\infty)^k.$$

We also define

$$H_r := H_r^1(\mathbb{R}^n, \mathbb{R}^k), \quad M_\sigma^r := M_\sigma \cap H_r$$

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where, by definition,  $u \in H_r^1(\mathbb{R}^n, \mathbb{R}^k)$  if  $u \in H^1(\mathbb{R}^n, \mathbb{R}^k)$  and

$$u^j(x) = u^j(y) \text{ if } |x| = |y|, \text{ a.e.}$$

for every  $1 \leq j \leq k$ . On the Hilbert spaces  $H$  and  $H_r$ , we consider the norm induced by the scalar product

$$(u, v)_H := \sum_{j=1}^k (u_j, v_j)_{H^1}.$$

Solutions to (E) with the variational characterisation above are interesting by several means: critical points of  $E$  over  $M_\sigma$  correspond to standing-wave  $k$ -uples solutions to the system of non-linear Klein-Gordon equations

$$(k\text{-NLKG}) \quad \partial_{tt}u_j - \Delta_x u_j + m_j^2 u_j + \partial_{z_j} G(u) = 0, \quad 1 \leq j \leq k$$

through the map

$$(1) \quad (u, \omega) \mapsto (e^{-i\omega_1 t} u_1(x), \dots, e^{-i\omega_k t} u_k(x)).$$

Secondly, if we denote  $H^1(\mathbb{R}^n, \mathbb{C}^k) \times L^2(\mathbb{R}^n, \mathbb{C}^k)$  by  $X$ , on solutions to ( $k$ -NLKG) the quantities

$$(\text{Energy}) \quad \mathbf{E}: X \rightarrow \mathbb{R},$$

$$(\phi, \phi_t) \mapsto \frac{1}{2} \int_{\mathbb{R}^n} |D\phi|^2 + |\phi_t|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^k (m_j^2 \phi_j^2 + 2k^{-1} G(\phi))$$

$$(\text{Charges}) \quad \mathbf{C}_j: X \rightarrow \mathbb{R},$$

$$(\phi, \phi_t) \mapsto -\text{Im} \int_{\mathbb{R}^n} \overline{\phi_j} \phi_t^j.$$

are constant (under the assumption  $G(u) = G(|u_1|, \dots, |u_k|)$ ) and

$$E(u, \omega) = \mathbf{E}(u_1, \dots, u_k, -i\omega_1 u_1, \dots, -i\omega_k u_k)$$

$$C(u, \omega) = \mathbf{C}(u_1, \dots, u_k, -i\omega_1 u_1, \dots, -i\omega_k u_k).$$

Such equalities turned out to be crucial to prove the orbital stability of standing-wave solutions to the scalar NLKG in [2], and to a coupled NLKG in [8]. Finally, according to [3], solutions  $v$  to the scalar NLKG with initial datum  $\Phi \in X$  such that the energy/charge ratio

$$\Lambda(\Phi) := \frac{\mathbf{E}(\Phi)}{m\mathbf{C}(\Phi)} < 1$$

have a non-dispersive property. We do not address in this work the orbital stability or dispersion.

We use the notation

$$m := m_1, \quad H_r^* := H_r \setminus 0, \quad \Sigma_* := \Sigma \setminus 0$$

and assume that  $G$  is continuously differentiable and

$$(A_0) \quad G(z) = G(|z_1|, \dots, |z_k|);$$

$$(A_1) \quad F(z) := G(z) + \frac{1}{2} \sum_{j=1}^k m_j^2 z_j^2 \geq 0, \quad G(0) = 0;$$

$$(A_2) \quad |DG(z)| \leq c(|z|^{p-1} + |z|^{q-1}), \quad 2 < p \leq q < \frac{2n}{n-2};$$

$$(A_3) \quad \alpha := \inf_{z \in \Sigma_*} \frac{F(z)}{|z|^2} < \frac{m^2}{2};$$

for every  $1 \leq j \leq k$

$$(A_4) \quad \alpha_j := \inf_{\substack{\sum_{h \neq j} z_h^2 \neq 0}} \frac{F(z)}{\sum_{h \neq j} z_h^2} > \alpha.$$

Under the assumptions above, we can prove the following

**Theorem (Main).** *There exists an open subset  $\Omega \subset (0, +\infty)^k$  such that the infimum of  $E$  is achieved on  $M_\sigma^r$  for every  $\sigma \in \Omega$ .*

The technique we use is similar to the one adopted in [4] in the scalar case  $k = 1$ . Therein it is showed that if a minimising sequence  $(u_n, \omega_n)$  for  $E$  over  $M_\sigma^r$  is such that  $\omega_n \rightarrow \omega < m$ , then a subsequence of  $(u_n)$  converges on  $H^1$ . The existence of such sequences is provided by the inequality

$$(2) \quad \inf_{H_r^* \times \Sigma_*} \Lambda < \inf_{H_r^* \times \Sigma_*^m} \Lambda$$

where

$$\Lambda(u, \omega) := \frac{E(u, \omega)}{C(u, \omega)}$$

and

$$\Sigma_*^m = \Sigma_* \cap \{z \geq m\}.$$

In higher dimension,  $\Sigma_*^m$  should be replaced by

$$\Sigma_*^{\mathbf{m}} := \bigcup_{j=1}^k \Sigma_*^{m_j}$$

where

$$\Sigma_*^{m_j} = \{z \in \Sigma_* \mid z_j \geq m_j\}.$$

A direct attempt to prove the inequality (2) lead to minimise  $\Lambda(u, \cdot)$  over the set  $\Sigma_*^{\mathbf{m}}$ , whose boundary consists of  $3^k - 1$  pieces each of them leading to a different condition on the non-linear term  $F$ . We believe that all these conditions include (A<sub>4</sub>).

So, rather than proving (2), we show in Lemma **Coercive** that when  $\Lambda(u_n, \omega_n)$  converges to its infimum, each component of  $\omega_n$  converges to  $\sqrt{2\alpha} < m$ .

1. PROPERTIES OF THE FUNCTIONAL  $E$ 

We recall some properties of the functional  $E$ . We include the proof of them only for the sake of completeness, as they are similar to the scalar case [2].

**Proposition 1.** *Suppose that  $G$  fulfils the assumptions  $(A_1)$  and  $(A_2)$ . Then,  $E$  is continuously differentiable; if  $\sigma \in (0, +\infty)^k$ , then  $E$  is coercive on  $M_\sigma$ .*

*Proof.* The continuity and the differentiability of  $E$  follows from analogous techniques used in theorems on bounded domains as [1, Theorem 2.2 and 2.6, p.16,17]. For a detailed proof we also refer to [8, Proposition 2].

Let  $(u, \omega) \in M_\sigma$  and set  $E = E(u, \omega)$ . By  $(A_2)$ , we have

$$(3) \quad \omega_i \leq \frac{2E}{\sigma_i}, \quad \|Du\|_{L^2}^2 \leq 2E.$$

By  $(A_2)$  there exists  $\varepsilon > 0$  such that

$$(4) \quad F(u) \geq m^2|u|^2/4, \text{ if } |u| \leq \varepsilon.$$

We have

$$E \geq \int_{|u| \geq \varepsilon} F(u) + \int_{|u| < \varepsilon} F(u).$$

From (4), it follows that

$$(5) \quad \|u\|_{L^2(|u| < \varepsilon)}^2 \leq 4E/m^2.$$

On the other hand, by the Sobolev inequality

$$(6) \quad \begin{aligned} \int_{|u| \geq \varepsilon} |u|^2 &= \varepsilon^{2-2^*} \int_{|u| \geq \varepsilon} \varepsilon^{2^*-2} |u|^2 \leq \varepsilon^{2-2^*} \int_{|u| \geq \varepsilon} |u|^{2^*} \\ &\leq c^{2^*} \varepsilon^{2-2^*} \|Du\|_{L^2}^{2^*} \end{aligned}$$

where  $c$  is the constant in the proof of [6, Théorème IX.9, p. 165]. From (5) and (6)

$$\|u\|_{L^2}^2 \leq \frac{4E}{m^2} + 2c^{2^*} \varepsilon^{2-2^*} E.$$

Along with (3), we obtained that the sub-levels of  $E$  are bounded, then  $E$  is coercive.  $\square$

Hereafter, we assume that  $\sigma_j > 0$  for every  $1 \leq j \leq k$ .

**Proposition 2.** *Let  $(u_n, \omega_n) \subset M_\sigma^r$  be a Palais-Smale sequence and  $\omega_n \rightarrow \omega$  such that  $\omega_i < m_i$ . Then  $(u_n)$  has a converging subsequence.*

*Proof.* By Proposition 1,  $(u_n)$  is bounded. Thus, by [5, Theorem A.I'], we can suppose that

$$(7) \quad u_n^j \rightharpoonup u_j \text{ in } H_r^1, \quad u_n^j \rightarrow u_j \text{ in } L^p \cap L^q$$

for every  $1 \leq j \leq k$ . Because  $(u_n, \omega_n)$  is a Palais-Smale sequence, there are

$$(\lambda_n) \subset \mathbb{R}, \quad (v_n, \eta_n) \subset H_r^* \times \mathbb{R}^k$$

such that

$$(8) \quad DE(u_n, \omega_n) = \sum_{j=1}^k \lambda_n^j DC_j(u_n, \omega_n) + (v_n, \eta_n), \quad (v_n, \eta_n) \rightarrow 0.$$

We multiply (8) by  $(0, e_j) \in \{0\} \times \mathbb{R}^k$  and obtain

$$\omega_n^j \|u_n^j\|_{L^2}^2 = \lambda_n^j \|u_n^j\|_{L^2}^2 + \eta_n^j$$

whence

$$(9) \quad \lambda_n^j = \omega_n^j - \frac{\eta_n^j \omega_n^j}{\sigma_j}.$$

We multiply (8) by  $(\phi, 0) \in H_r \times \{0\}$  and obtain

$$\begin{aligned} \sum_{j=1}^k (Du_n^j, D\phi_j)_{L^2} + m_j^2 (u_n^j, \phi_j)_{L^2} + \int_{\mathbb{R}^n} DG(u_n) \cdot \phi \\ + \sum_{j=1}^k (\omega_n^j)^2 (u_n^j, \phi_j)_{L^2} - 2 \sum_{j=1}^k \lambda_n^j \omega_n^j (u_n^j, \phi_j)_{L^2} = (v_n, \phi)_H \end{aligned}$$

which, by (9), becomes

$$\begin{aligned} (10) \quad \sum_{j=1}^k (Du_n^j, D\phi_j)_{L^2} + m_j^2 (u_n^j, \phi_j)_{L^2} + \int_{\mathbb{R}^n} DG(u_n) \cdot \phi \\ - \sum_{j=1}^k (\omega_n^j)^2 (u_n^j, \phi_j)_{L^2} = (v_n, \phi)_H - 2 \sum_{j=1}^k \frac{\eta_n^j (\omega_n^j)^2}{\sigma_j} (u_n^j, \phi_j)_{L^2}. \end{aligned}$$

From (A<sub>1</sub>), (10) can be written as

$$\begin{aligned} (11) \quad \sum_{j=1}^k (Du_n^j, D\phi_j)_{L^2} + (m_j^2 - \omega_n^j)^2 (u_n^j, \phi_j)_{L^2} \\ = (v_n, \phi)_H - \sum_{j=1}^k \beta_n^j (u_n^j, \phi_j)_{L^2} \\ - \int_{\mathbb{R}^n} (DG(u_n) - DG(u)) \cdot (u_n - u) \end{aligned}$$

where

$$(12) \quad \beta_n^j := \left( \omega_n^j - (\omega_n^j)^2 - \frac{2\eta_n^j (\omega_n^j)^2}{\sigma_j} \right) \rightarrow 0, \quad 1 \leq j \leq k.$$

Given a pair of integers  $(n, m)$ , taking the difference of the equations, (11<sub>n</sub>) and (11<sub>m</sub>) with  $\phi = u_n - u_m$ , we obtain

$$(13) \quad \begin{aligned} & \sum_{j=1}^k \|Du_n^j - Du_m^j\|_{L^2}^2 + (m_j^2 - \omega_j^2 + \beta_n^j + \beta_m^j) \|u_n^j - u_m^j\|_{L^2}^2 \\ &= (v_n - v_m, u_n - u_m)_H - \int_{\mathbb{R}^n} \left( DG(u_n) - DG(u_m) \right) \cdot (u_n - u_m). \end{aligned}$$

Thus from the assumption  $\omega_j < m_j$  and (12), there exists  $c_0 > 0$  such that

$$(14) \quad \begin{aligned} \|u_n - u_m\|_H^2 &\leq c_0 \sum_{j=1}^k \left( \|Du_n^j - Du_m^j\|_{L^2}^2 \right. \\ &\quad \left. + (m_j^2 - \omega_j^2 + \beta_n^j + \beta_m^j) \|u_n^j - u_m^j\|_{L^2}^2 \right) \end{aligned}$$

and

$$(15) \quad (v_n - v_m, u_n - u_m)_H \leq \|u_n - u_m\|_H (\gamma_n + \gamma_m)$$

where

$$(16) \quad \gamma_n := \|v_n\|_H \rightarrow 0.$$

We have

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \left( DG(u_n) - DG(u_m) \right) \cdot (u_n - u_m) \right| \\ & \leq \sum_{j=1}^k \left( \|u_n^j - u_m^j\|_{L^p} + \|u_n^j - u_m^j\|_{L^q} \right) \end{aligned}$$

it is convenient to estimate each of the two summand of the inequality above as follows: by [6, Corollaire IX.10, p. 165]

$$(17) \quad \begin{aligned} \|u_n^j - u_m^j\|_{L^p} &= \|u_n^j - u_m^j\|_{L^p}^{1/2} \|u_n^j - u_m^j\|_{L^p}^{1/2} \\ &\leq \|u_n^j - u_m^j\|_{L^p}^{1/2} \|u_n^j - u_m^j\|_{H^1}^{1/2} \\ &\leq \delta_{n,m}^p \|u_n^j - u_m^j\|_{H^1}^{1/2} \end{aligned}$$

where

$$\delta_{n,m}^p := \max_{1 \leq j \leq k} \|u_n^j - u_m^j\|_{L^p}^{1/2}$$

is infinitesimal by (7). By the Hölder inequality, we have

$$\sum_{j=1}^k \|u_n^j - u_m^j\|_{H^1}^{1/2} \leq k^{4/3} \|u_n - u_m\|_H^{1/4}$$

whence

$$(18) \quad \begin{aligned} & \left| \int_{\mathbb{R}^n} \left( DG(u_n) - DG(u_m) \right) \cdot (u_n - u_m) \right| \\ & \leq k^{4/3} (\delta_{n,m}^p + \delta_{n,m}^q) \|u_n - u_m\|_H^{1/4}. \end{aligned}$$

Now, putting together (14, 15, 18) we obtain

$$(c_0)^{-1} \|u_n - u_m\|_H^{7/8} \leq c_1(\gamma_n + \gamma_m) + \delta_n + \delta_m$$

where

$$c_1 = \sup_{n,m} \left( \sum_{j=1}^k \|u_n^j - u_m^j\|_{H^1}^2 \right)^{3/4}.$$

Then each of  $(u_n^j)$  is a Cauchy sequence in  $H^1$  for every  $1 \leq j \leq k$ , thus converges to  $v_j \in H^1$ . From (7),  $v_j = u_j$ , thus  $u_n \rightarrow u$  in  $H_r$ .  $\square$

## 2. PROPERTIES OF $\Lambda$

We define the following energy/charge ratio

$$\Lambda(u, \omega) := \frac{E(u, \omega)}{\sum_{j=1}^k C_j(u, \omega)}$$

and introduce the notation

$$a(u) := \frac{1}{2} \int_{\mathbb{R}^n} |Du|^2 + \int_{\mathbb{R}^n} F(u), \quad b_j(u) := \int_{\mathbb{R}^n} u_j^2.$$

If we fix  $u \in H_*$ , we have the smooth function defined on  $\Sigma_*$

$$\Lambda(u, \cdot) : \Sigma \rightarrow \mathbb{R}, \quad \omega \mapsto \Lambda(u, \omega) = \frac{1}{2} \cdot \frac{2a(u) + \sum_{j=1}^k b_j(u) \omega_j^2}{\sum_{j=1}^k b_j(u) \omega_j}$$

It is not hard to check, arguing by induction on  $k$ , that the following properties hold for  $\Lambda(u, \cdot)$ :

- (i) is non-negative and achieves its infimum in a (unique) interior point lying on the principal diagonal. We denote this point by  $\omega(u)$  and each of its components by  $\xi(u)$ ;
- (ii) there holds

$$\Lambda(u, \omega(u)) = \xi(u), \quad \xi(u)^2 = \frac{2a(u)}{\sum_{j=1}^k b_j(u)}.$$

**Proposition 3.**  $\inf_{H_*} \xi = \sqrt{2\alpha}$ .

*Proof.* That the right member is not greater than the left one, follows from the definition of  $\alpha$ . In fact,

$$\xi(u)^2 = \frac{\int_{\mathbb{R}^n} |Du|^2 + 2 \int_{\mathbb{R}^n} F(u)}{\sum_{j=1}^k b_j(u)} \geq \frac{\int_{\mathbb{R}^n} |Du|^2 + 2\alpha \int_{\mathbb{R}^n} |u|^2}{\int_{\mathbb{R}^n} |u|^2} \geq 2\alpha,$$

where in the last inequality we neglected the gradient terms. In order to prove the opposite inequality, we define

$$u_R(x) = \begin{cases} z & \text{if } |x| \leq R \\ (1 + R - |x|)z & \text{if } R \leq |x| \leq R + 1 \\ 0 & \text{if } |x| \geq R + 1. \end{cases}$$

where  $z \in \Sigma_*$  is an arbitrary point and  $R > 0$ . We compute its gradient

$$Du_R^j(x) = \begin{cases} 0 & \text{if } |x| \leq R \text{ or } |x| \geq R+1 \\ -\frac{z_j x}{|x|} & \text{otherwise.} \end{cases}$$

By standard computations, we have

$$\begin{aligned} \|u_R^j\|_{L^2}^2 &= \mu(B_1)R^n z_j^2 + O(R^{n-1}) \\ \int_{\mathbb{R}^n} F(u_R) &= \mu(B_1)R^n F(z) + O(R^{n-1}), \\ \|Du_R^j\|_{L^2}^2 &= O(R^{n-2}), \end{aligned}$$

where  $B_1$  is the unit ball of  $\mathbb{R}^n$  and  $\mu(B_1)$  is its Lebesgue measure. Then,

$$\xi(u_R)^2 = \frac{2\mu(B_1)R^n F(z) + o(R^n)}{\mu(B_1)R^n |z|^2 + o(R^n)} = o(1) + \frac{2F(z)}{|z|^2}.$$

Taking the limit as  $R \rightarrow +\infty$ , we obtain

$$\inf_{H_*} \xi^2 \leq \frac{2F(z)}{|z|^2}$$

for ever  $z \in \Sigma_*$ . Because  $z$  was chosen arbitrarily, we obtain the conclusion.  $\square$

Looking at the behaviour of  $\Lambda(u, \cdot)$ , one can easily deduce that sequences converging to the minimum value converge to the minimum point. The next lemma exploits the uniform behaviour of  $\Lambda$  on  $u$ .

**Lemma (Coercive).** *For every  $\varepsilon > 0$  there exists  $\eta$  such that*

$$\Lambda(u, \omega) < \sqrt{2\alpha} + \eta$$

*implies*

$$|\omega_j - \sqrt{2\alpha}| < \varepsilon$$

*Proof.* For every  $1 \leq j \leq k$  and  $u \in H_*$ , we define

$$B_j(u) = \frac{b_j(u)}{\sum_{j=1}^k b_j(u)}.$$

We divide the proof in three steps.

*Step 1.* We show that if  $k \geq 2$  and  $\eta$  is small enough, there exists  $\delta_0 \in (0, 1)$  such that

$$(19) \quad B_j(u) \in (\delta_0, 1 - \delta_0).$$

It is useful to define  $\alpha_* := \min\{\alpha_j \mid 1 \leq j \leq k\}$ . Due to (A<sub>4</sub>) we have  $\alpha < \alpha_*$ . By property (ii) of  $\Lambda$

$$(20) \quad \sqrt{2\alpha} + \eta > \Lambda(u, \omega) \geq \xi(u);$$



we fix  $1 \leq j \leq k$ . We have

$$\begin{aligned}\xi(u)^2 &= \frac{\|Du\|_{L^2}^2 + 2 \int_{\mathbb{R}^n} F(u)}{\sum_{j=1}^k b_j(u)} \\ &= \frac{\|Du\|_{L^2}^2 + 2 \int_{\mathbb{R}^n} F(u)}{\sum_{j \neq s} b_j(u)} \cdot \frac{1}{1 + B_j(u)} \geq \frac{2\alpha_j}{1 + B_j(u)}\end{aligned}$$

where in the last inequality we neglected the gradient terms and used the notation of the assumption (A<sub>4</sub>). From (20) and the inequality above, we obtain

$$\sqrt{2\alpha} + \eta > \frac{\sqrt{2\alpha_j}}{\sqrt{1 + B_j(u)}}$$

whence

$$(21) \quad B_j(u) > \frac{2\alpha_j}{(\sqrt{2\alpha} + \eta)^2} - 1 \geq \frac{2\alpha_*}{(\sqrt{2\alpha} + \eta)^2} - 1 =: \delta_0.$$

Thus, if  $\delta_0 > 0$ , then obtain a bound from below for  $B_j(u)$ . Thus, we require

$$(22) \quad \eta < \sqrt{2\alpha_*} - \sqrt{2\alpha}$$

which gives  $B_j(u) > \delta_0$  for every  $1 \leq j \leq k$ . Because

$$\sum_{j=1}^k B_j(u) = 1$$

it follows that

$$B_j(u) = 1 - \sum_{h \neq j} B_h(u) \leq 1 - (k-1)\delta_0 \leq 1 - \delta_0.$$

*Step 2.* If  $\Lambda(u, \omega) < \sqrt{2\alpha} + \eta$ , then  $\omega$  is bounded from above. If  $\eta$  is chosen as in (22) and  $k \geq 2$  then

$$\Lambda(u, \omega) \geq \frac{\sum_{j=1}^k B_j \omega_j^2}{2 \sum_{j=1}^k B_j \omega_j} \geq \frac{\delta_0}{2(1 - \delta_0)} \cdot \frac{\sum_j \omega_j^2}{\sum_j \omega_j}.$$

Thus,

$$\sum_{j=1}^k \omega_j^2 \leq 2C_0 \cdot \sum_{j=1}^k \omega_j$$

where

$$C_0 := \frac{(\sqrt{2\alpha} + \eta)(1 - \delta_0)}{\delta_0}$$

Thus,

$$(23) \quad \omega_j < C_0(1 + \sqrt{k}), \quad 1 \leq j \leq k.$$

When  $k = 1$ ,

$$\sqrt{2\alpha} + \eta > \Lambda(u, \omega) \geq \omega/2$$

thus,

$$(24) \quad \omega < 2(\sqrt{2\alpha} + \eta).$$

*Step 3.* We conclude the proof of the lemma. When  $k \geq 2$ ,

$$\begin{aligned} \eta &\geq \Lambda(u, \omega) - \Lambda(u, \omega(u)) = \Lambda(u, \omega) - \xi(u) \\ &= \frac{1}{2} \left( \frac{\xi^2 + \sum_{j=1}^k B_j \omega_j^2 - 2 \sum_{j=1}^k B_j \omega_j \xi}{\sum_{j=1}^k B_j \omega_j} \right) \\ &= \frac{1}{2} \frac{\sum_{j=1}^k B_j (\omega_j - \xi)^2}{\sum_{j=1}^k B_j \omega_j} = \frac{1}{2} \sum_{j=1}^k \left( \frac{B_j}{\sum_{j=1}^k B_j \omega_j} \right) \cdot (\omega_j - \xi)^2 \\ &\geq \frac{\delta_0}{2(1 - \delta_0)C_0(\sqrt{k} + 1)} \sum_{j=1}^k (\omega_j - \xi)^2 \end{aligned}$$

the last inequality follows from the bounds on  $\omega$  (23) and on  $B_j$  from *Step 1* and *Step 2*. Thus,

$$\frac{2\eta(1 - \delta_0)^2(\sqrt{k} + 1)(\sqrt{2\alpha} + \eta)}{\delta_0^2} > (\omega_j - \xi)^2.$$

Because  $\xi < \sqrt{2\alpha} + \eta$ ,

$$(25) \quad |\omega_j - \sqrt{2\alpha}| < \sqrt{\eta} \left( \sqrt{\eta} + \frac{1 - \delta_0}{\delta_0} \cdot \left( 2(\sqrt{2\alpha} + \eta)(\sqrt{k} + 1) \right)^{1/2} \right)$$

for every  $1 \leq j \leq k$ . Because the term on the right member of the inequality above is  $O(\sqrt{\eta})$ , the proof is complete when  $k \geq 2$ . When  $k = 1$ , by (24)

$$\eta > \Lambda(u, \omega) - \xi(u) = \frac{1}{2\omega}(\omega - \xi)^2 \geq \frac{1}{4(\sqrt{2\alpha} + \eta)}(\omega - \xi)^2$$

then

$$|\omega - \xi| < 2 \left( \eta(\sqrt{2\alpha} + \eta) \right)^{1/2}$$

whence

$$(26) \quad |\omega - \sqrt{2\alpha}| < \sqrt{\eta} \left( \sqrt{\eta} + 2 \left( \sqrt{2\alpha} + \eta \right)^{1/2} \right)$$

□

*Proof of the Theorem Main.* Let  $(u', \omega')$  be such that

$$\Lambda(u', \omega') < \sqrt{2\alpha} + \eta$$

where  $\eta$  is chosen in such a way that the right term in (25) (for  $k \geq 2$ ) or (26) (when  $k = 1$ ) is not greater than

$$\frac{1}{2}(m - \sqrt{2\alpha}).$$

We define

$$\sigma_j := \omega'_j \int_{\mathbb{R}^n} (u'_j)^2.$$

Clearly  $(u', \omega') \in M_\sigma^r$ . Now, let us take a minimising sequence  $(u_n, \omega_n)$  of  $E$  over  $M_\sigma^r$ . By the Ekeland's theorem [11, Theorem 5.1, p. 48], we can suppose that  $(u_n, \omega_n)$  is a Palais-Smale sequence. Then, there exists  $n_0 \in \mathbb{N}$  such that

$$\Lambda(u_n, \omega_n) \leq \Lambda(u', \omega') = \Lambda(u', \omega') < \sqrt{2\alpha} + \eta.$$

if  $n \geq n_0$ . Thus

$$\Lambda(u_n, \omega_n) < \sqrt{2\alpha} + \eta, \quad n \geq n_0.$$

By the preceding lemma, we have

$$|\omega_n^j - \sqrt{2\alpha}| < \frac{1}{2}(m - \sqrt{2\alpha});$$

up extract a subsequence from  $(\omega_n^j)$ , we can suppose that each of the  $(\omega_n^j)$  converge to some  $\omega_j$ . Therefore

$$m - \omega_j = m - \sqrt{2\alpha} + \sqrt{2\alpha} - \omega_j \geq \frac{1}{2}(m - \sqrt{2\alpha}) > 0.$$

By Proposition 2, we obtain that  $E$  achieves its infimum on  $M_\sigma$ . Finally, we observe that the subset of  $(0, +\infty)^k$

$$\Omega := \left\{ \sigma \in (0, +\infty)^k \mid \frac{I(\sigma)}{\sum_{j=1}^k \sigma_j} < \sqrt{2\alpha} + \eta \right\}$$

is open. In fact, let  $\sigma_0 \in \Omega$  and  $(u_0, \omega_0)$  be a minimiser of  $E$  over  $M_{\sigma_0}$ . Thus,

$$\Lambda(u_0, \omega_0) < \sqrt{2\alpha} + \eta.$$

Given an arbitrary  $\sigma$ , we define

$$\omega_\sigma^j := \frac{\omega_0^j \sigma_j}{\sigma_0^j}.$$

Using the continuity of  $\Lambda$  on  $\omega$ , it can be showed that

$$\Lambda(u_0, \omega_\sigma) = \Lambda(u_0, \omega_0) + O(|\sigma - \sigma_0|).$$

Thus, if  $|\sigma - \sigma_0|$  is small enough,

$$\Lambda(u, \omega_\sigma) < \sqrt{2\alpha} + \eta$$

which concludes the proof.  $\square$

**Corollary.** *There exists  $\eta_0$  such that, for every  $\eta < \eta_0$  there exists  $(u_\eta, \omega_\eta)$  such that  $u_\eta$  is a solution to (E)*

$$\Lambda(u_\eta, \omega_\eta) < \sqrt{2\alpha} + \eta, \quad \omega_\eta^j - \sqrt{2\alpha} < \eta$$

for every  $1 \leq j \leq k$ .

*Proof.* The existence of  $(u_\eta, \omega_\eta)$  follow from Theorem [Main](#). All we need to prove is that  $u_\eta > 0$  and solves the elliptic system in [\(E\)](#). So, let  $\sigma \in (0, \infty)^k$  be as in Theorem [Main](#) and  $(u, \omega) \in M_\sigma^r$  a minimiser of  $E$  over  $M_\sigma^r$ . From [\(A<sub>0</sub>\)](#),

$$(v, \omega) := (|u_1|, \dots, |u_k|, \omega)$$

is also a minimiser of  $E$  over  $M_\sigma^r$  and, thus, a constrained critical point. There is a natural action of the orthogonal group  $O(n, \mathbb{R})$  on  $H^1(\mathbb{R}^n, \mathbb{R}^k)$

$$\begin{aligned} O(n) \times H^1(\mathbb{R}^n, \mathbb{R}^k) \times [0, +\infty)^k &\rightarrow H^1(\mathbb{R}^n, \mathbb{R}^k) \times [0, +\infty)^k \\ (G, u, \omega) &\mapsto G \cdot (u, \omega) := (u(Gx), \omega) \end{aligned}$$

this action restricts to  $M_\sigma$  and the set of fixed point is  $M_\sigma^r$ . Moreover,  $E$  is invariant for the action

$$E(u, \omega) = E(u(Gx), \omega).$$

By the symmetric criticality principle [[10](#), §0],  $(u, \omega)$  is a critical point of  $E$  over  $M_\sigma$ . Thus, each of the equations in [\(E\)](#) can be written as

$$(27) \quad -\Delta v_j + c_j(x)v_j = 0$$

where

$$(28) \quad c_j(x) = \begin{cases} m_j^2 - \omega_j^2 + \frac{\partial_{z_j} G(v)}{v_j} & \text{if } v_j(x) \neq 0 \\ m_j^2 - \omega_j^2 & \text{if } v_j(x) = 0. \end{cases}$$

From [\(A<sub>2</sub>\)](#)

$$(29) \quad |c_j(x)| \leq m_j^2 - \omega_j^2 + c(|v_j|^{p-2} + |v_j|^{q-2}).$$

Thus, for every bounded domain  $V \subset \mathbb{R}^n$ ,  $c_j \in L^\infty(V)$ , because  $v_j$  is continuous. Then, we can apply the maximum principle to the elliptic equation [\(27\)](#) (for example, [[7](#), Lemma 1, p. 556]) and conclude that  $v_j > 0$  on  $V$ . Because this holds for every  $V$ ,  $v_j > 0$  on  $\mathbb{R}^n$ . Hence  $u_\eta$  has a sign for every  $\eta$ . Up to adjusting the signs of  $u_\eta^j$ ,  $(u, \omega)$  is the sought solution to [\(E\)](#).  $\square$

Some remarks are in order.

*Concentration of minimising sequences.* If we add the requirement

$$(A_5) \quad \int_{\mathbb{R}^n} F(u_1^*, \dots, u_k^*) \leq \int_{\mathbb{R}^n} F(u)$$

where  $u_j^*$  denotes the decreasing rearrangement of  $u_j$ , then minimisers of  $E$  over  $M_\sigma^r$  are minimisers of  $E$  over  $M_\sigma$ . We define

$$I(\sigma) := \inf_{M_\sigma} E.$$

Moreover, if for every minimiser  $(u, \omega)$  there holds

$$(A_6) \quad \overline{\lim} E(u_1(\cdot + y_n^1), u_2(\cdot + y_n^2), \dots, u_k(\cdot + y_n^k), \omega) > E(u, \omega)$$

if  $|y_n^j - y_n^h|$  is not bounded for some  $j \neq h$ , then it is natural to expect the *sub-additivity* property of  $I$ , that is

$$I(\sigma) < I(\sigma') + I(\sigma - \sigma')$$

for every  $\sigma'$  such that  $\sigma' \neq \sigma$  and  $\sigma'_j \leq \sigma$  for every  $1 \leq j \leq k$ . Thus, by means of the concentration-compactness Lemma, it would follow that a minimising sequence exhibits a concentration behaviour.

*Some example of non-linearity.* It might be surprising the fact that in our solutions all the frequencies tend to converge in the interval  $(\sqrt{2\alpha}, m)$  regardless of the relations between  $m_j$  and  $m_h$  for  $j \neq h$ . This follows from the assumption  $(A_4)$ : when the non-linearity  $G$  does not have coupling terms, that is

$$(30) \quad G(z) = G_1(z_1) + \cdots + G_k(z_k)$$

the system (E) reduces to  $k$  scalar elliptic equations

$$-\Delta u_j + (m_j^2 - \omega_j^2)u_j + G'_j(u_j) = 0$$

each of them can be solved separately as in [4] or [2] in order to obtain positive solutions. By the Derrick-Pohozaev identity and the maximum principle it follows

$$m_j > \omega_j > \sqrt{2\alpha_j}, \quad 1 \leq j \leq k.$$

So, if  $G$  is as in (30), the frequencies  $\omega_j$  have a different behaviour from the one proved in Theorem Main, where

$$\sqrt{2\alpha} < \omega_j < m \leq m_j, \quad 1 \leq j \leq k.$$

In fact, a non-linearity as in (30) does not satisfy the assumption  $(A_4)$ : given  $z \neq (0, \dots, 0)$ , we have

$$\frac{F(z)}{|z|^2} \geq \frac{\sum_{j=1}^k \alpha_j |z_j|^2}{|z|^2} \geq \min_{1 \leq j \leq k} \alpha_j.$$

Also, it is more simple to treat each equation of the case (30) separately, using the result of [2] or the theorem when  $k = 1$ .

Some non-linearities  $G$  satisfying assumptions  $(A_1-A_4)$  are given by

$$G(z) = -z_1^p z_2^p + |z|^q, \quad z \in \Sigma$$

$$G(z) := G(|z_1|, |z_2|)$$

when  $k = 2, N = 3$  and

$$1 < p, \quad 2p < q < 5.$$

When  $k = 3, N = 3$ , we can define

$$G(z) = -(z_1 z_2)^{p_1} - (z_2 z_3)^{p_2} - (z_1 z_3)^{p_3} - (z_1 z_2 z_3)^{p_4} + |z|^q, \quad z \in \Sigma$$

$$G(z) := G(|z_1|, |z_2|, |z_3|).$$

and

$$\begin{aligned} 2 < 2p_i < q < 5, \text{ for } 1 \leq i \leq 3 \\ 3 < 3p_4 < q. \end{aligned}$$

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